Lecture 23 - Addendum

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1 Proofs of the main results from the lecture

Let's recall the definition from one of the previous lectures.

Definition 1.1. *Function $f(a_1, a_2, \ldots, a_m)$ is called **multilinear** if it is linear in every argument, i.e. for any i*

$$f(a_1, \ldots, a_i' + a_i'', \ldots, a_m) = f(a_1, \ldots, a_i', \ldots, a_m) + f(a_1, \ldots, a_i'', \ldots, a_m)$$
$$f(a_1, \ldots, \lambda a_i, \ldots, a_m) = \lambda f(a_1, \ldots, a_i, \ldots, a_m).$$

Definition 1.2. *Multilinear function $f(a_1, a_2, \ldots, a_m)$ is called **alternating** if it changes the sign after interchanging any 2 arguments, i.e. for any i and j*

$$f(a_1, \ldots, a_i, \ldots, a_j, \ldots, a_m) = -f(a_1, \ldots, a_j, \ldots, a_i, \ldots, a_m).$$

If f is alternating, then it is equal to 0 if any 2 arguments are equal. It is true, because if we interchange these 2 arguments, the function will not change, but from the other hand, it should change its sign. So, it is equal to 0.

Now we're able to formulate the main result about alternating multilinear functions.

Theorem 1.3. *For any $c \in \mathbb{R}$ in the vector space $\mathbb{R}^n$ there exists the unique alternating multilinear function f, such that*

$$f(e_1, e_2, \ldots, e_n) = c \tag{1}$$

(where e_i's are rows with 1 on i-th place, and 0's on all other places). Moreover, this function is equal to

$$f(a_1, a_2, \ldots, a_n) = c \cdot \sum_{\text{all permutations of } n \text{ elements } \sigma} \operatorname{sgn}(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \tag{2}$$

$$= c \cdot \sum_{(k_1, k_2, \ldots, k_n)} \operatorname{sgn}(k_1, k_2, \ldots, k_n) a_{1k_1} a_{2k_2} \cdots a_{nk_n}, \tag{3}$$

where a_{ik} is the k-th component of the row a_i, and the summation is over all permutations of numbers from 1 to n.

Proof. 1. Let f be an alternating multilinear function, such that $f(e_1, \dots, e_n) = c$. Then

$$f(a_1, a_2, \dots, a_n) = f\left(\sum_{k_1} a_{1k_1} e_{k_1}, \sum_{k_2} a_{2k_2} e_{k_2}, \dots, \sum_{k_n} a_{nk_n} e_{k_n}\right)$$
$$= \sum_{k_1, k_2, \dots, k_n} a_{1k_1} a_{2k_2} \cdots a_{nk_n} f(e_1, e_2, \dots, e_n).$$

Since f is alternating, if any 2 numbers from $k_1, k_2, \dots, k_n$ are equal, then $f(e_{k_1}, e_{k_2}, \dots, e_{k_n}) = 0$. If all of them are different, then

$$f(e_{k_1}, e_{k_2}, \dots, e_{k_n}) = c \operatorname{sgn}(k_1, k_2, \dots, k_n).$$

Let's prove it. If this equality is true for some permutation, then it is true for any other permutation which we can get from the initial by transposition of any 2 elements (since after transposition both left-hand side and right-hand side change signs). But this equality is true for identity permutation, and since it is possible to get any permutation from identity, then this equality is true for all permutations. So, we get that f satisfies the expression (3). So, if f satisfying the given conditions exists, then it has the form (3) and so it is unique.

2. Now we'll prove that the function f given by the formula (3) is alternating multilinear, and satisfy the condition (1). Linearity by any argument is obvious, since for any i the equality (3) can be written as

$$f(a_1, a_2, \dots, a_n) = \sum_j a_{ij} u_j,$$

where u_j's do not depend on a_i. The condition (1) also holds, since in the expression for $f(e_1, e_2, \dots, e_n)$ all summands except the term, corresponding to the identity permutation are equal to 0, and the term, corresponding to the identity permutation is equal to 1. Now we should check that this function is alternating.

Let we interchange arguments a_i and a_j. Then all permutations can be divided into pairs different only by interchanging k_i and k_j. Terms from these pairs are included in the expression (3) with different signs (since one of them is different from the other by one transposition). After interchanging a_i and a_j they change their roles, and so the whole expression changes its sign.

$\square$

If $c = 1$ we will denote such function by det.

Definition 1.4. *The* ***determinant*** *of the square $n \times n$-matrix $A = (a_{ij})$ is*

$$\det A = \det(a_1, a_2, \dots, a_n),$$

where $a_1, a_2, \dots, a_n$ are rows of A.

Corollary 1.5. *If f is an arbitrary alternating multilinear function of rows of the matrix, then*

$$f(A) = f(I) \det A,$$

where I is the identity matrix.

Moreover, we proved that $\det A^\top = \det A$, so we have the following corollary:

Corollary 1.6. *The determinant of the matrix is the alternating multilinear function of its columns.*

Now we're able to prove the main theorems about properties of determinants.

Theorem 1.7 (Determinant of the product). *For any square matrices A and B*

$$\det(AB) = \det A \det B.$$

Proof. It's easy to see that rows $c_1, c_2, \ldots, c_n$ of the matrix AB can be obtained from the rows $a_1, \ldots, a_n$ of the matrix A by multiplication by B:

$$c_i = a_i B \qquad i = 1, \ldots, n.$$

So, if the matrix B is fixed, then the determinant of AB is the alternating multilinear function of the rows of A: let, without loss of generality $a_1 = a_1' + a_1''$, where a_1' and a_1'' are arbitrary rows. Then

$$\begin{aligned}\det(AB) = \det(a_1 B, a_2 B, \ldots, a_n B) &= \det((a_1' + a_1'')B, a_2 B, \ldots, a_n B) \\ &= \det(a_1' B + a_1'' B, a_2 B, \ldots, a_n B) \\ &= \det(a_1' B, a_2 B, \ldots, a_n B) + \det(a_1'' B, a_2 B, \ldots, a_n B).\end{aligned}$$

Other properties can be checked in the same manner. Thus, if B is fixed, $\det AB$ is the alternating multilinear function. So, by the corollary 1.5 we have:

$$\det AB = \det IB \cdot \det A = \det B \cdot \det A = \det A \cdot \det B.$$

$\square$

Theorem 1.8 (The determinant of the block matrix). *Let*

$$A = \begin{pmatrix} B & D \\ 0 & C \end{pmatrix},$$

where B and C are square matrices. Then

$$\det A = \det B \det C.$$

Proof. If matrices B and D are fixed, then the determinant of A is the alternating multilinear function of its last rows, and so it is the alternating multilinear function of rows of the matrix C. Thus, by the corollary 1.5

$$\det A = \det \begin{pmatrix} B & D \\ 0 & I \end{pmatrix} \cdot \det C.$$

If the matrix D is fixed, then the first multiplicand is the alternating multilinear function of rows of B, and so

$$\det \begin{pmatrix} B & D \\ 0 & I \end{pmatrix} = \det \begin{pmatrix} I & D \\ 0 & I \end{pmatrix} \cdot \det B = \det B$$

(because $\begin{pmatrix} I & D \\ 0 & I \end{pmatrix}$ is triangular with 1's on its diagonal). So,

$$\det A = \det B \det C$$

$\square$